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Families of isospectral matrix Hamiltonians by deformation of the Clifford algebra on a phase space

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Abstract
By using a recently developed method, we report five different families of isospectral 2×2 matrix Hamiltonians defined on a four-dimensional (4D) phase space. The employed method is based on a realization of the supersymmetry idea on the phase space whose complexified Clifford algebra structure is deformed with the Moyal star-product. Each reported family comprises many physically relevant special models. 2D Pauli Hamiltonians, systems involving spin–orbit interactions such as Aharonov–Casher-type systems, a supermembrane toy model and models describing motion in noncentral electromagnetic fields as well as Rashba- and Dresselhaus-type systems from semiconductor physics are obtained, together with their super-partners, as special cases. A large family of isospectral systems characterized by the whole set of analytic functions is also presented.

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1. Introduction

The supersymmetry (SUSY) idea, originally introduced in the relativistic field theory to study the models in which boson and fermion become indistinguishable, has also provided powerful methods in quantum mechanics (QM) to identify as well as to construct new isospectral pairs of systems [1–3]. It is the usual Schrödinger formulation of QM where the overwhelming majority of applications using the supersymmetric methods have taken place. Deformation quantization has become the third autonomous and logically complete formulation of QM beyond the conventional ones based on the Schrödinger formulation and path-integral formulation. However, the first study illustrating the utility of Moyal $*$-product in realizing supersymmetric techniques on a two-dimensional (2D) classical phase space has appeared only a decade ago [4]. More recent studies [5–7] combining the $*$-product and

Clifford product make it possible to include the fermionic degrees of freedom in the context of deformation quantization.

The main goal of deformation quantization, which is also known as the phase-space formulation of QM and also as Moyal quantization [8, 9], is to carry out all quantum calculations in the classical phase space of Hamiltonian mechanics (for an extensive chronological list of references we refer to [9]). In this quantization scheme all quantum effects are encoded in the associative but non-commutative $*$-product with respect to which all real-valued phase-space functions become quantum observables. A remarkable property of this composition rule, not shared by the usual operator product, is that it enables one to do all calculations to any desired order of the Planck constant $\hbar$. This also makes it possible to easily identify the classical limit of any evolution computed by means of $*$-product. In [7], a realization of the SUSY methods on a 4D phase space was achieved by deforming the complexified Clifford algebra $\mathbf{C}_4(\mathbb{C})$ of the space with the Moyal $*$-product in composing the components of Clifford forms. The resulting associative product, denoted by $*_{\mathrm{MC}}$, is called the Moyal–Clifford (MC) product. This becomes the $*$-product of matrix-valued functions when a matrix representation of the algebra is used. The method developed in [7] by means of MC-algebra extends the applicability of deformation quantization to almost all area of QM.

Our main aim in the present study is to report five different families of isospectral pair of 2×2 matrix Hamiltonians defined on a 4D phase space. These are constructed by the method of [7] in which the phase space is endowed with the fermionic degrees of freedom. Each reported family contains some arbitrary, or partially constrained, phase-space functions which enable the family to be large enough to comprise many special pairs. By this method, many model Hamiltonians which are the main subject of active researches in different branches of physics, ranging from supermembrane theory to the semiconductor physics, can be identified together with their super-partners as special cases. The basic points of this method are outlined below where our main notation and conventions are also fixed. The crucial role of the Clifford algebra [10] in these constructions and the general structure of underlying SUSY algebra are given in appendix A.

The main goal of [7] was to construct two isospectral matrix Hamiltonians

$$H_j = H_* \mathbf{1} + H_{jF}, \qquad j = 1, 2, \tag{1}$$

having a common bosonic part $H_* \mathbf{1}$ with

$$H_* = \frac{1}{2} \sum_{j=1}^{2} (P_j * P_j + W_j * W_j),$$

but different fermionic parts H_{jF} given by

$$\begin{aligned} H_{1F} &= \frac{\mathrm{i}}{2} B_+ \sigma_3, \\ H_{2F} &= \frac{\mathrm{i}}{2} B_- \sigma_3 - \mathrm{i}[W_2, P_1]_M \sigma_1 - \mathrm{i}[W_1, W_2]_M \sigma_2, \\ B_\pm &= [W_1, P_1]_M \pm [W_2, P_2]_M. \end{aligned} \tag{2}$$

Here W_j and P_j are real-valued functions of the canonical coordinates $(\mathbf{q}, \mathbf{p})$, σ_k's are the usual Pauli matrices, $\mathbf{1}$ stands for the unit 2×2 matrix and $[W, P]_M = W * P - P * W$ denotes the Moyal bracket corresponding to the $*$-product:

$$* = \exp \left[\frac{1}{2} \mathrm{i}\hbar \sum_{j=1}^{2} \left(\overleftarrow{\partial}_{q_j} \overrightarrow{\partial}_{p_j} - \overleftarrow{\partial}_{p_j} \overrightarrow{\partial}_{q_j} \right) \right],$$

where an arrow over $\partial_x (=\partial/\partial x)$ indicates the direction for its action. The isospectral property of H_1 and H_2 is evident from the following double intertwining relations:

$$L_2 *_{\mathrm{MC}} H_1 = H_2 *_{\mathrm{MC}} L_2, \qquad L_1 *_{\mathrm{MC}} H_2 = H_1 *_{\mathrm{MC}} L_1, \tag{3}$$

where L_j's denote the matrix-valued intertwining functions

$$L_1 = 2\mathrm{i}\begin{pmatrix} 0 & 0 \\ C_1 & -C_2 \end{pmatrix}, \qquad L_2 = 2\mathrm{i}\begin{pmatrix} C_2 & 0 \\ C_1 & 0 \end{pmatrix}, \tag{4}$$

with $\sqrt{2}C_j = W_j + \mathrm{i}P_j$. As is shown in appendix A (for more details see [7]) relations (3) hold if and only if C_1 and C_2 are Moyal commutating. This is equivalent to the following two conditions:

$$[W_1, W_2]_M = [P_1, P_2]_M, \qquad [W_1, P_2]_M = [W_2, P_1]_M. \tag{5}$$

A direct check of relations (3) and (5) may seem a bit lengthly; however, we should emphasize the surprising role of the Clifford algebra in this regard and recommend the reader to have a look at the appendix before checking the above equations.

Our classification of isospectral families is entirely based on physically relevant particular solutions of the algebraic conditions (5). In fact, they constitute two (countably) infinite sets of partial differential equations each arising from the equality of the coefficients of equal powers of $\hbar$ in both sides of each condition. Evidently, their general solutions seem to be impossible. Nevertheless, as will be shown below, even their particular solutions comprise physically of interest many isospectral pair of systems from different branches of physics. In this regard, physical relevance considerably eases the investigation.

As the first two families, considered together in the next section, we present the cases in which the above conditions are satisfied from the outset by appropriate choices of the involved functions. We recover two 2D Pauli Hamiltonians which are supersymmetric partners and have SUSY structures by themselves in the first family. In the second family one of the partners is supersymmetric by itself. In this family a realization of the spin–orbit coupling on the phase space which allows us to identify Aharonov–Casher (AC)-type systems as special cases is presented. As is shown in section 3, some variants of a supermembrane toy model and some Hamiltonians describing motions in noncentral electromagnetic fields can be recognized as special cases of the third family. This is realized when there is only one condition arising from (5). In the fourth family of section 4, conditions (5) go over to the well-known Cauchy–Riemann conditions. A salient feature of this family is that to each analytic function an isospectral pair of Hamiltonians is associated in such a way that one of them is purely bosonic and the ferminonic part of the other indicates a Zeeman-type interaction with a purely gradient magnetic field. The fifth family of section 5 consists of isospectral pairs containing a 2D Rashba- or Dresselhaus-type system both of which are well known in the semiconductor spintronics and spin Hall effect. Concluding remarks are given in section 6.

2. The first two families of Hamiltonians

There are two different ways of getting rid of conditions (5) by suitable choices of the phase-space functions. As presented below, each of such a choice leads to a different family of Hamiltonians.

2.1. 2D Pauli Hamiltonians

The simplest way of fulfilling conditions (5) from the beginning is to choose $C_1 = 0$, that is, $P_1 = 0 = W_1$. Then for

$$W_2 = \frac{1}{\sqrt{M}}\left(p_1 - \frac{e}{c}A_1\right), \qquad P_2 = \frac{1}{\sqrt{M}}\left(p_2 - \frac{e}{c}A_2\right) \tag{6}$$

such that A_j's depend only on q_j's we obtain

$$[W_2, P_2]_M = \mathrm{i}\frac{e\hbar}{Mc}B(\mathbf{q}). \tag{7}$$

Considering A_j's as the components of a vector potential, $B(\mathbf{q}) = \partial_{q_1}A_2 - \partial_{q_2}A_1$ represents the corresponding inhomogeneous magnetic field perpendicular to the q_1q_2-plane. Then in terms of

$$H_* = \frac{1}{2M}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2, \tag{8}$$

and $\gamma = e\hbar/2Mc$ we have

$$H_1 = H_*\mathbf{1} - \gamma B(\mathbf{q})\sigma_3, \qquad H_2 = H_*\mathbf{1} + \gamma B(\mathbf{q})\sigma_3. \tag{9}$$

These are 2D Pauli Hamiltonians which differ from each other by the sign in front of the Zeeman term $-\boldsymbol{\mu}\cdot\mathbf{B} = -\gamma B\sigma_3$. The Hamiltonians are related not by a charge conjugation $e \to -e$ but by the reflection $B \to -B$.

It is well known that both H_1 and H_2 are supersymmetric in the usual formulation of QM [1–3]. Here we have established this fact in the phase-space formulation and have proved that they are supersymmetric partners of each other. In this case the intertwining functions and factorized forms of H_j are as follows:

$$\begin{aligned}
L_1 &= -2\mathrm{i}C_2\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \qquad L_2 = 2\mathrm{i}C_2\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\\
H_1 &= \begin{pmatrix}\bar{C}_2 * C_2 & 0\\ 0 & C_2 * \bar{C}_2\end{pmatrix} = \{Q_1, Q_1^\dagger\}_{\mathrm{MC}},\\
H_2 &= \begin{pmatrix}C_2 * \bar{C}_2 & 0\\ 0 & \bar{C}_2 * C_2\end{pmatrix} = \{Q_2, Q_2^\dagger\}_{\mathrm{MC}}.
\end{aligned} \tag{10}$$

Here the complex supercharges are defined by

$$Q_1 = 2\bar{C}_2\sigma_+, \qquad Q_2 = 2C_2\sigma_+, \tag{11}$$

where $2\sigma_\pm = \sigma_1 \pm \mathrm{i}\sigma_2$.

2.2. The second family of Hamiltonians

Another simple way for the fulfillment of conditions (5) is to choose $C_2 = kC_1$, where k is any nonzero complex number. For $k = 0$, that is, for $W_2 = 0 = P_2$ we have the trivial case in which both H_1 and H_2 are equal. In fact, for this equality it is sufficient to take one of W_2 and P_2 to be zero. Two illustrative examples can be given for nonzero k. In the first case we take k to be a nonzero real constant such that $W_2 = kW_1$ and $P_2 = kP_1$. Hence,

$$\begin{aligned}
H_* &= \frac{1+k^2}{2}(P_1 * P_1 + W_1 * W_1),\\
H_1 &= H_*\mathbf{1} + \mathrm{i}\frac{1+k^2}{2}[W_1, P_1]_M\sigma_3,\\
H_2 &= H_*\mathbf{1} + \mathrm{i}[W_1, P_1]_M\left(\frac{1-k^2}{2}\sigma_3 - k\sigma_1\right).
\end{aligned} \tag{12}$$

As the second example we take $k = \mathrm{i}\ell$, where ℓ is a nonzero real constant. Then $W_2 = -\ell P_1$, $P_2 = \ell W_1$ and we obtain the same expressions as (12) provided that k is replaced with ℓ and $k\sigma_1$ in the third term of H_2 is replaced with $-\ell\sigma_2$. For $k = 1$ the intertwining functions and the Hamiltonians read

$$
\begin{aligned}
L_1 &= 2\mathrm{i}C_1\begin{pmatrix}0 & 0\\ 1 & -1\end{pmatrix}, \qquad L_2 = 2\mathrm{i}C_1\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix},\\
H_1 &= 2\begin{pmatrix}\bar{C}_1 * C_1 & 0\\ 0 & C_1 * \bar{C}_1\end{pmatrix},\\
H_2 &= \begin{pmatrix}\{C_1,\bar{C}_1\}_M & [C_1,\bar{C}_1]_M\\ [C_1,\bar{C}_1]_M & \{C_1,\bar{C}_1\}_M\end{pmatrix}.
\end{aligned}
\tag{13}
$$

In terms of the complex supercharge $Q_1 = \sqrt{2}\bar{C}_1\sigma_+$ we have

$$
H_1 = \{Q_1, Q_1^\dagger\}_{\mathrm{MC}}, \qquad [H_1, Q_1]_{\mathrm{MC}} = 0 = Q_1 *_{\mathrm{MC}} Q_1,
\tag{14}
$$

where and in (13) $\{,\}_M$ and $(\{,\}_{\mathrm{MC}})$ denote the anti-Moyal and anti-MC brackets. Evidently H_1 is supersymmetric and remains so for the two cases considered above by taking constant multiple of Q_1. As a result, this family is characterized by two arbitrary real-valued functions W_1 and P_1 as well as by a nonzero complex constant.

2.3. Spin–orbit interactions on a phase space and AC-type systems

The AC effect [11] is known as the electromagnetic dual of the well-known Aharonov–Bohm effect [12, 13]. The former describes the behavior of neutral particles with magnetic moment $\boldsymbol{\mu} = \mu\boldsymbol{\sigma}$ aligned perpendicular to their plane of motion (here taken as the q_1q_2-plane) under the influence of a static electric field $\mathbf{E} = (E_1, E_2)$ of an impenetrable line-charge also aligned perpendicular to the plane [14]. Although it can be described relativistically, the AC effect is mostly discussed in the nonrelativistic limit. In this case, for the motion in the charge-free $(\boldsymbol{\nabla}\cdot\mathbf{E} = 0)$ region, the AC Hamiltonian operator $\hat{H}_{\mathrm{AC}}$ can be written in the following three equivalent ways:

$$
\begin{aligned}
2M\hat{H}_{\mathrm{AC}} &= \hat{Q}\hat{Q}^\dagger,\\
&= (\hat{\mathbf{p}} - \mathbf{E}\times\boldsymbol{\mu})^2 - \mu^2E^2\mathbf{1},\\
&= (\hat{\mathbf{p}}^2 + \mu^2E^2)\mathbf{1} + \mu\big[\mathrm{i}\hbar(\boldsymbol{\nabla}\times\mathbf{E})_3 + 2(\mathbf{E}\times\hat{\mathbf{p}})_3\big]\sigma_3.
\end{aligned}
\tag{15}
$$

All these expressions hold in the usual Schrödinger formulation, $\hat{\mathbf{p}} = -\mathrm{i}\hbar\boldsymbol{\nabla}$ is the usual momentum operator and $\hat{Q} = \boldsymbol{\sigma}\cdot(\hat{\mathbf{p}} - \mathrm{i}\mu\mathbf{E})$. We should refer to [12] for the second expression of (15) and note that the terms in the square bracket of the third line are Hermitian only if taken together. For central static fields we have

$$
\boldsymbol{\nabla}\times\mathbf{E} = 0, \qquad \mathbf{E} = -\frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}\mathbf{r},
$$

where $\mathbf{r} = (q_1, q_2)$ and $r = \left(q_1^2 + q_2^2\right)^{1/2}$. We then obtain, from the third line of (15),

$$
\hat{H}_{\mathrm{AC}} = \frac{1}{2M}(\hat{\mathbf{p}}^2 + \mu^2E^2)\mathbf{1} - \frac{\mu}{Mr}\frac{\mathrm{d}V}{\mathrm{d}r}(\mathbf{r}\times\hat{\mathbf{p}})_3\sigma_3,
$$

with the usual form of the spin–orbit (SO) interaction in the last term.

We will now show that various forms of SO-couplings, and hence, many AC types of isospectral pairs can be realized on the phase space. For this purpose, in terms of two arbitrary functions f and g of r and $J_3 = q_1p_2 - q_2p_1$ we consider

$$W_1 = g(r)\mathbf{r}\cdot\mathbf{p} = g(r) * (\mathbf{r}\cdot\mathbf{p}) - \frac{\mathrm{i}\hbar}{2} r g'(r), \tag{16}$$

$$P_1 = f(r) J_3 = f(r) * J_3. \tag{17}$$

Here $g'(r)$ denotes the derivative of g with respect to its argument which, when there is no risk of confusion, will be suppressed. J_3 is the component of the angular momentum vector generating rotations in the $q_1 q_2$-plane. Since f, g, $\mathbf{r}\cdot\mathbf{p}$ and W_1 are scalar under such rotations, they Moyal commute with J_3. But

$$[\mathbf{r}\cdot\mathbf{p}, f(r)]_M = -\mathrm{i}\hbar r f'(r) \tag{18}$$

implies

$$[W_1, P_1]_M = -\mathrm{i}\hbar r g(r) f'(r) J_3. \tag{19}$$

It is also straightforward to verify that

$$\begin{aligned} (\mathbf{r}\cdot\mathbf{p}) * (\mathbf{r}\cdot\mathbf{p}) &= (\mathbf{r}\cdot\mathbf{p})^2 + \frac{\hbar^2}{2}, \\ J_3 * J_3 &= J_3^2 - \frac{\hbar^2}{2}, \\ P_1 * P_1 &= f^2\left(J_3^2 - \frac{\hbar^2}{2}\right). \end{aligned} \tag{20}$$

On the other hand, the computation of the $*$-square of W_1 is not so easy. By virtue of (16) and (18) we first compute

$$W_1 * W_1 = g^2 * (\mathbf{r}\cdot\mathbf{p}) * (\mathbf{r}\cdot\mathbf{p}) - 2\mathrm{i}\hbar (r g g') * (\mathbf{r}\cdot\mathbf{p}) - \frac{\hbar^2}{2} r \left(g g' + r g g'' + \frac{r}{2} g'^2 \right). \tag{21}$$

By longer computations which require going up to the $\hbar^2$ terms in the expansion of the $*$-product, we obtain

$$\begin{aligned} g^2 * (\mathbf{r}\cdot\mathbf{p})^2 &= g^2(\mathbf{r}\cdot\mathbf{p})^2 + 2\mathrm{i}\hbar r g g'(\mathbf{r}\cdot\mathbf{p}) - \frac{\hbar^2}{4} r^2 (g^2)'', \\ (r g g') * (\mathbf{r}\cdot\mathbf{p}) &= r g g'(\mathbf{r}\cdot\mathbf{p}) + \frac{\mathrm{i}\hbar}{2} r (g g' + r g g'' + r g'^2). \end{aligned}$$

On substituting these relations into (21), in view of the first relation of (20), we arrive at

$$W_1 * W_1 = g^2(\mathbf{r}\cdot\mathbf{p})^2 + g^2\frac{\hbar^2}{2} + \frac{\hbar^2}{2} r g' \left(g + \frac{r}{2} g' \right). \tag{22}$$

Adding (22) and the third relation of (20) together yields from (12)

$$H_* = \frac{1+k^2}{2}\left[(g r p)^2 + \frac{\hbar^2}{2} r g' \left(g + \frac{1}{2} r g' \right) \right], \tag{23}$$

where we have taken $f^2 = g^2$ and made use of

$$J_3^2 + (\mathbf{r}\cdot\mathbf{p})^2 = r^2 p^2. \tag{24}$$

If in (19) and (23) we take $g = a/r$, $f = \epsilon a/r$, where $\epsilon = \pm 1$ and

$$M^{-1} = a^2(1+k^2), \tag{25}$$

we obtain from the last two relations of (12) and from (23)

$$
\begin{aligned}
H_* &= \frac{1}{2M}\left(p^2 - \frac{\hbar^2}{4r^2}\right),\\
H_{1F} &= -\epsilon\frac{\hbar}{2Mr^2}J_3\sigma_3,\\
H_{2F} &= -\epsilon\frac{\hbar}{2Mr^2}J_3\left(\frac{1-k^2}{1+k^2}\sigma_3 - \frac{2k}{1+k^2}\sigma_1\right).
\end{aligned}
\tag{26}
$$

By comparing with (15) and with the usual form of the SO-interaction we see that H_1 is of the AC type with the electric field of a uniform line-charge

$$\mathbf{E} = -\epsilon\frac{\hbar}{2\mu r^2}\mathbf{r},$$

and with the induced electric dipole energy $-\hbar^2/(4r^2)$, where the minus sign implies that the induced electric dipole moment is in the same direction as $\mathbf{E}$. Note also that there are infinitely many super-partners H_2's whose fermionic parts are indexed by k or by the angle $\vartheta \in (0, 2\pi)$ such that

$$\cos\vartheta = \frac{1-k^2}{1+k^2}, \qquad \sin\vartheta = \frac{2k}{1+k^2}.$$

Finally in this section we should note that a larger family can be generated by taking

$$g = \frac{a}{r}, \qquad f = g\kappa(r),$$

and by rewriting (22), by virtue of (24), as

$$W_1 * W_1 = a^2\left[p^2 - \frac{1}{r^2}\left(J_3^2 - \frac{\hbar^2}{4}\right)\right].$$

These lead us to

$$H_* = \frac{1}{2M}\left[p^2 + \frac{\hbar^2}{4r^2}(1-2\kappa^2) - \frac{J_3^2}{r^2}(1-\kappa^2)\right],$$

with H_{1F} and H_{2F} given as in (26) provided that the factor $-\epsilon/r^2$ is replaced with $(\kappa/r)'$. This new family contains an arbitrary function $\kappa(r)$ which reduces to (26) for $\kappa = \epsilon$. In all these cases J_3 is a constant of motion.

3. Third family of Hamiltonians

By adopting the cartesian coordinates for q_j's we now consider

$$P_1 = \frac{p_x}{\sqrt{M}}, \qquad P_2 = \frac{p_y}{\sqrt{M}}, \qquad W_j = W_j(x, y), \qquad j = 1, 2,$$

for which the first condition of (5) is identically satisfied and the second one yields

$$\partial_y W_1 = \partial_x W_2. \tag{27}$$

Then we have $H_* = (1/2M)p^2 + V$ with $2V = W_1^2 + W_2^2$ and

$$
\begin{aligned}
B_\pm &= \mathrm{i}\frac{\hbar}{\sqrt{M}}(\partial_x W_1 \pm \partial_y W_2),\\
H_1 &= H_*\mathbf{1} - \frac{\hbar}{2\sqrt{M}}(\partial_x W_1 + \partial_y W_2)\sigma_3,\\
H_2 &= H_*\mathbf{1} - \frac{\hbar}{2\sqrt{M}}(\partial_x W_1 - \partial_y W_2)\sigma_3 + \frac{\hbar}{\sqrt{M}}(\partial_x W_2)\sigma_1.
\end{aligned}
\tag{28}
$$

In terms of $g = x^2 - 4ax + b$ and $f = ay + c$, where a, b and c are some real constants, the choices

$$W_1 = \tfrac{1}{4}(y^2 - g), \qquad W_2 = \tfrac{1}{2}xy + f,$$

satisfy (27) and the fermionic parts of the above Hamiltonians take the forms

$$H_{1F} = -\frac{\hbar}{\sqrt{M}}a\sigma_3, \qquad H_{2F} = \frac{\hbar}{2\sqrt{M}}(x\sigma_3 + y\sigma_1). \tag{29}$$

For $b = 0 = c$ we have

$$V = \frac{1}{2}\left[\frac{1}{16}r^4 + a^2r^2 + \frac{1}{2}ax(3y^2 - x^2)\right], \tag{30}$$

which reduces to the potential energy function of a quartic oscillator for $a = 0$ and H_1 becomes pure bosonic.

3.1. A supermembrane toy model

H_{2F} given by (29) is, for $2\sqrt{M} = \hbar$, the same as the fermionic part of

$$\hat{H}_{tm} = (p^2 + x^2y^2)\mathbf{1} + x\sigma_3 + y\sigma_1, \tag{31}$$

but the potential energy functions of H_2 and $\hat{H}_{tm}$ are different. The model described by $\hat{H}_{tm}$ has served as a toy model for a certain class of supersymmetric matrix models and has been largely discussed within the context of reductions of supersymmetric Yang–Mills theories, of regulated theories of supermembranes and M-theory [15, 16]. With the supercharge

$$\hat{Q}_{tm} = \hat{p}_x\sigma_3 - \hat{p}_y\sigma_1 - xy\sigma_2,$$

and parity operator $\hat{P}$ acting on a 2×1 column spinor $\Psi(x, y)$ as

$$(\hat{P}\Psi)(x, y) = \tfrac{1}{\sqrt{2}}(\sigma_1 + \sigma_3)\Psi(y, x),$$

the set $\{\hat{H}_{tm}, \hat{Q}_{tm}, \hat{P}\}$ exhibits the following SUSY algebra structure:

$$\hat{H}_{tm} = \hat{Q}_{tm}^2, \qquad \hat{P}^2 = \mathbf{1}, \qquad \{\hat{Q}_{tm}, \hat{P}\} = 0,$$

in the usual Schrödinger formulation of QM. Here the products are the usual operator products and $\{,\}$ is the corresponding anti-commutator.

We first note that the supersymmetric structure of this model is preserved if everything is understood in the phase-space context of this paper. Secondly, by changing the coefficient function xy of σ_2 in the supercharge operator with an arbitrary function $g(x, y)$ such that

$$Q = p_x\sigma_3 - p_y\sigma_1 - g\sigma_2,$$

we obtain a generalization of $\hat{H}_{tm}$ in the phase space with $V = g^2$ and with the new fermionic part

$$\mathrm{i}([p_y, g]_M\sigma_3 + [p_x, g]_M\sigma_1) = \hbar(\partial_y g)\sigma_3 + \hbar(\partial_x g)\sigma_1,$$

which reduces to the fermionic part of (31) for $g = xy$. For $g = r^2/4$ we can rewrite H_2 as

$$2H_2 = Q *_{\mathrm{MC}} Q, \qquad P^2 = \mathbf{1}, \qquad \{Q, P\}_{\mathrm{MC}} = 0, \tag{32}$$

which emphasize its supersymmetric structure. This corresponds taking $a = 0$ in (29) and (30) and hence H_1 becomes a pure bosonic Hamiltonian with a quartic oscillator potential. Equations (32) imply that P, which is the phase-space version of the so-called Witten operator $\hat{P}$, commutes in the MC sense with H_2.

3.2. Motion in 2D *noncentral fields*

For $W_1 = -\mu D_2/\sqrt{M}$ and $W_2 = \mu D_1/\sqrt{M}$, condition (27) reads $\nabla \cdot \mathbf{D} = 0$ and from (28) we get

$$H_1 = H_* \mathbf{1} + \frac{\mu\hbar}{2M}(\nabla \times \mathbf{D})_3 \sigma_3, \tag{33}$$

$$H_* = \frac{1}{2M}(p^2 + \mu^2 D^2). \tag{34}$$

If $\mathbf{D}$ is identified with a 2D noncentral electric or magnetic field, then H_1 describes the motion of an uncharged and polarized magnetic moment μ in such a field. In this case $\mu^2 D^2$ corresponds to the energy of an induced moment and the condition $\nabla \cdot \mathbf{D} = 0$ implies that, as in the AC effect, the motion takes place in the charge-free region for the identification $\mathbf{D} \to \mathbf{E}$. For identification with a magnetic field $\nabla \cdot \mathbf{D} = 0$ is one of the Maxwell equations.

The super-partner of H_1 is found, from (28), to be

$$H_2 = H_* + \frac{\mu\hbar}{2M}[(\partial_x D_2 + \partial_y D_1)\sigma_3 + 2\partial_x D_1 \sigma_1]. \tag{35}$$

As anticipated above for

$$D_1 = \tfrac{1}{2}xy + ay, \qquad D_2 = \tfrac{1}{4}(x^2 - y^2) - ax,$$

the fermionic part of H_2 is the same, up to a constant multiple, as the fermionic part of (31). This observation makes it possible to consider some variants of a supermembrane toy model as the super-partners of a genuine physical system such as the motion of an uncharged and polarized magnetic moment in a 2D noncentral electromagnetic field.

4. Isospectral Hamiltonians by analytic functions

In this section we will prove that to each analytic function of two real variables there corresponds a pair of isospectral 2×2 matrix Hamiltonians. For this fourth family we choose

$$W_1 = p_x, \qquad P_1 = p_y, \qquad W_2 = u(x, y), \qquad P_2 = v(x, y), \tag{36}$$

for which conditions (5) yield

$$\partial_x u = \partial_y v, \qquad \partial_y u = -\partial_x v. \tag{37}$$

These are the well-known Cauchy–Riemann conditions for $C_2 = (u + \mathrm{i}v)/\sqrt{2}$ to be an analytic function. Evidently, u and v are harmonic functions of cartesian coordinates x, y, that is, they belong to the kernel of 2D Laplace operator $\nabla^2 : \nabla^2 u = 0 = \nabla^2 v$. In this case $B_\pm$ vanish and

$$\begin{aligned} H_1 &= H_* \mathbf{1} = \left[\tfrac{1}{2}p^2 + V(x, y)\right]\mathbf{1}, \\ H_2 &= H_* \mathbf{1} + \hbar[(\partial_y u)\sigma_1 - (\partial_x u)\sigma_2], \\ V &= \tfrac{1}{2}|C_2|^2 = \tfrac{1}{4}(u^2 + v^2). \end{aligned} \tag{38}$$

Note that H_1 is pure bosonic and the fermionic part of H_2 is of the form

$$H_{2F} = \hbar(\boldsymbol{\sigma} \times \boldsymbol{\nabla} u)_3 = -\hbar\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} v.$$

Equations (38) explicitly reveal the fact that to each analytic function of two variables there corresponds an isospectral pair of 2×2 matrix Hamiltonians. This infinite family can be characterized in the following way: a pure bosonic Hamiltonian whose potential is the square modulus of an analytic function is supersymmetric partner when two fermionic degrees of freedom interacting with the bosonic part as in $-\hbar\boldsymbol{\sigma}\cdot\nabla v$ are added to it. This represents a Zeeman-type interaction corresponding to the inhomogeneous magnetic field which is a gradient (hence curl-free) field lying in the plane of motion. Note that H_1 and the bosonic part of H_2 depend only on the square modulus of the chosen analytic function and the fermionic part of H_2 is sensitive to its local phase.

Although many physical examples can be given, we shall present two examples: one is in terms of cartesian coordinates and the other accepts separation of variables in the plane polar coordinates (r,θ). For the first example we choose

$$u = a(x^2-y^2) + bxy + cx + ky + \ell,$$
$$v = -\tfrac{1}{2}b(x^2-y^2) + 2axy - kx + cy + m,$$

where a, b, c, k, ℓ and m are some real constants. By choosing $\ell = 0 = m$ and then by retaining only two of the remaining constants to be different from zero we obtain

$$V = \begin{cases} \frac{1}{2}\left(a^2+\frac{b^2}{4}\right)r^4, \\ \frac{1}{8}[b^2r^2+4k(k+bx)]r^2, \\ \frac{1}{2}(c^2+k^2)r^2. \end{cases} \tag{39}$$

The first is 2D quartic oscillator, the last one is the usual 2D isotropic oscillator and the second one is a nonseparable potential. As the second example we take

$$u = ar^{k+1}\sin(k+1)\theta, \qquad v = -ar^{k+1}\cos(k+1)\theta,$$

which specify the potential as $V = a^2r^{2k+2}/2$ and

$$H_2 = H_1 + \hbar a(k+1)r^k[\cos(k\theta)\sigma_1 - \sin(k\theta)\sigma_2]. \tag{40}$$

This family can be extended to include systems describing planar motion under the influence of electromagnetic fields. Indeed, if W_1 and P_1 are replaced by

$$W_1 = \frac{1}{\sqrt{M}}\left(p_x - \frac{e}{c}A_x\right), \qquad P_1 = \frac{1}{\sqrt{M}}\left(p_y - \frac{e}{c}A_y\right),$$

without changing W_2 and P_2 given by (36), conditions (37) remain intact, but B_+ and B_- become equal to $(\mathrm{i}e\hbar/Mc)B(\mathbf{q})$. Here $B(\mathbf{q})$ is the associated inhomogeneous magnetic field perpendicular to the xy-plane of motion. Hence H_1 goes over to a Pauli-type Hamiltonian.

5. A family of Rashba- and Dresselhaus-type Hamiltonians

Useful analogs of a spin-1/2 system with its 2D state space are two-state atom and any two-state system for which Pauli matrices play a prominent role as well [17]. To uncover new isospectral systems in such a context, we rewrite H_{2F} as

$$H_{2F} = \frac{\mathrm{i}}{2}B_-\sigma_3 + \sqrt{2}(\mathcal{A}\sigma_+ + \bar{\mathcal{A}}\sigma_-), \tag{41}$$

$$\mathcal{A} = [W_2, \bar{C}_1]_M, \qquad \bar{\mathcal{A}} = -[W_2, C_1]_M. \tag{42}$$

In the families considered so far, the coefficient functions of $\sigma_\pm$ depended only on q_j's and hence they were Moyal commuting. Therefore, by requiring non-commutativity of these functions a completely new family can be generated. $\mathcal{A}$ and $\bar{\mathcal{A}}$ are the phase-space analogs of the bosonic raising and lowering operators when their Moyal commutator is a nonzero real constant. It has already been shown in [7] that when $[\mathcal{A}, \bar{\mathcal{A}}]_M = \pm 1$, the last two terms of (41) represent a resonant (for +) and a non-resonant (for −) Jaynes–Cummings-type models which are well known in quantum optics [17]. Phase-space characteristics of these two systems, such as their spectra, eigen-spinors and related Wigner functions, were also computed in [7]. In fact, by allowing $[\mathcal{A}, \bar{\mathcal{A}}]_M$ to be proportional to the appropriate products of related lowering and raising functions, physically relevant many models can be generated. Since this family is elaborated to some extent in the mentioned work, it will not be pursued any further here.

As the fifth family, we shall consider the cases for which $\mathcal{A}$ (and hence $\bar{\mathcal{A}}$) is a function of momenta. Although in this case $\mathcal{A}$ and $\bar{\mathcal{A}}$ become Moyal commuting, we recover a new large family which is different from the previous ones. We shall present two special cases for which the last two terms of (41) become

$$H_R = \boldsymbol{\alpha}[(p_y + \mathrm{i}p_x)\sigma_+ + (p_y - \mathrm{i}p_x)\sigma_-] = \boldsymbol{\alpha}(\boldsymbol{\sigma} \times \mathbf{p})_3, \tag{43}$$

$$H_D = \beta[(p_x + \mathrm{i}p_y)\sigma_+ + (p_x - \mathrm{i}p_y)\sigma_-], \tag{44}$$

where α, β are real functions of $p = \left(p_x^2 + p_y^2\right)^{1/2}$ and we have chosen $\sqrt{2}\mathcal{A} = \alpha(p_y + \mathrm{i}p_x)$ for (43) and $\sqrt{2}\mathcal{A} = \beta(p_x + \mathrm{i}p_y)$ for (44). When α, β are constants, these are the well-known 2D Rashba [18] and Dresselhaus [19] Hamiltonians which have attracted a great deal of interest in recent research fields such as the semiconductor spintronics and spin Hall effect [20, 21]. They arise from different mechanisms of SO-couplings of electron spin to the electric fields in semiconductors and they play important role in studying electrical monitoring of spin and electrical detection of spin dynamics [22, 24]. Another remarkable property of these Hamiltonians is that they can be interpreted in the context of a Yang–Mills-type non-Abelian gauge theory. Indeed, in the presence of the usual kinetic energy term we can write, up to a constant multiple of **1**, the total Rashba Hamiltonian as [21, 23]

$$\tilde{H}_R = \frac{p^2}{2M} + H_R = \frac{1}{2M}\left[\left(p_x + \frac{1}{2}\theta\hbar\sigma_2\right)^2 + \left(p_y - \frac{1}{2}\theta\hbar\sigma_1\right)^2\right].$$

In this case, $\theta = 2M\alpha/\hbar$ can be regarded as a charge and $\hbar(-\sigma_2, \sigma_1)/2$ as the corresponding non-Abelian gauge potential such that Pauli matrices account for isospin-like degrees of freedom. In fact, as is apparent in the second line of (15), such an interpretation was made by Goldhaber, for the first time, for the AC-Hamiltonian [14].

Since H_R and H_D are related (for $\alpha = \beta$) to each other by the swap $(p_x, p_y) \to (p_y, p_x)$, from now on we concentrate only on the family related to H_R. The analysis given below can be carried out *mutatis mutandis* for the Dresselhaus case. Then from (5) and (42) we obtain

$$[W_2, W_1]_M = \mathrm{i}\alpha p_x = [P_2, P_1]_M, \qquad [W_2, P_1]_M = \mathrm{i}\alpha p_y = [W_1, P_2]_M. \tag{45}$$

Evidently, to each particular solution set $\{W_j, P_j\}$ of these conditions there corresponds an isospectral pair of systems such that one of them contains H_R in its fermionic part. As an illustrative example, we shall take $P_1 = p_x/\sqrt{M}$ and $W_1 = -p_y/\sqrt{M}$, for which conditions (45) amount to

$$-\partial_y W_2 = \eta p_x = \partial_x P_2, \qquad \partial_x W_2 = \eta p_y = \partial_y P_2,$$

where $\eta = \alpha\sqrt{M}/\hbar$. These can easily be integrated to find their general solutions as

$$W_2 = \eta J_3 + f, \qquad P_2 = \eta \mathbf{r} \cdot \mathbf{p} + g, \tag{46}$$

where f and g are arbitrary real-valued functions of p_x, p_y and (see section 2.3)

$$J_3 = xp_y - yp_x, \qquad \mathbf{r} \cdot \mathbf{p} = xp_x + yp_y.$$

It will be convenient to consider first the case where α is a constant. If we also require $B_- = 0$, we obtain $[W_2, P_2]_M = 0$ since W_1 and P_1 were chosen to be Moyal commuting. Hence, this requirement excludes the Zeeman-type interactions from both partner Hamiltonians. Recalling that $[J_3, \mathbf{r}\cdot\mathbf{p}]_M = 0$, the Moyal commutativity of W_2 and P_2 yields

$$(\mathbf{p}\times\nabla_p)_3 g = -\mathbf{p}\cdot\nabla_p f,$$

where ∇_p is the 2D gradient operator with respect to momentum variables. We now use (24), the first two relations of (20) and take $f = 0 = g$ to obtain $H_{2R} = H_1 + H_R$ which is isopectral with its bosonic part $H_1 = H_*\mathbf{1}$, where

$$H_* = \frac{p^2}{2M} + \frac{\eta^2}{2}\left[J_3^2 + (\mathbf{r}\cdot\mathbf{p})^2\right] = \frac{1}{2M}\left[1 + \left(\frac{\alpha M}{\hbar}\right)^2 r^2\right]p^2. \tag{47}$$

Here we made use of (24). In addition to the usual kinetic term, H_* contains a momentum-dependent potential energy term. The total Rashba Hamiltonian $\tilde{H}_R$ is an ideal model which relies on the fast motion of electron in the strong field of the nuclei (supposed at $r = 0$) and does not depend on position coordinates. In fact, $H_{2R} = H_*\mathbf{1} + H_R$ may be considered as a model which takes into account nonzero values of r and accepts $\tilde{H}_R$ in the limit $r \to 0$. Noting that J_3 is a constant of motion, H_* can be elaborated from different perspectives. For instance, one can impose the constraint $\mathbf{r}\cdot\mathbf{p} = 0$ which leads, up to a constant, to $\tilde{H}_R$. One can also interpret H_* as the kinetic term for a position-dependent mass

$$\frac{M}{1 + (\alpha M/\hbar)^2 r^2}.$$

When $\alpha/\hbar$ is small such that its square can be neglected, or alternatively, on the 2D phase-space plane $x = 0 = y$, we recover again the total Rashba Hamiltonian with its usual terms. When f and g are constants, H_* acquires the following additional terms:

$$\frac{\eta}{2}(f J_3 + g\mathbf{r}\cdot\mathbf{p}) + \frac{1}{2}(f^2 + g^2),$$

and when the constraint $B_- = 0$ is removed, H_1 and H_{2R} gain additional fermionic terms with different signs.

In the general $\alpha = \alpha(p)$ case, by similar calculations of section 2.3 we obtain for $f = 0 = g$:

$$\begin{aligned} W_2 * W_2 &= \eta^2\left(J_3^2 - \frac{\hbar^2}{2}\right), \\ P_2 * P_2 &= \eta^2(\mathbf{r}\cdot\mathbf{p})^2 + \frac{M}{2}\left[\alpha^2 + p\alpha'\left(\alpha + \frac{p}{2}\alpha'\right)\right], \\ B_\pm &= \pm[W_2, P_2]_M = \mp \mathrm{i}\frac{M}{\hbar}p\alpha\alpha' J_3. \end{aligned} \tag{48}$$

Hence

$$H_* = \frac{p^2}{2M} + \frac{M}{2\hbar^2}(\alpha r p)^2 + \frac{M}{4}p\alpha'\left(\alpha + \frac{p}{2}\alpha'\right),$$

$$H_{1F} = \frac{M}{2\hbar}p\alpha\alpha' J_3\sigma_3, \qquad H_{2F} = -H_{1F} + H_R,$$

and, as a particular case, for $\alpha = \hbar\omega/p$ where ω is a constant, we obtain

$$H_* = \frac{p^2}{2M} + \frac{1}{2}M\omega^2\left(r^2 - \frac{\hbar^2}{4p^2}\right), \qquad H_{1F} = -\frac{\hbar}{2}M\omega^2\frac{J_3\sigma_3}{p^2}. \tag{49}$$

H_{1F} indicates a SO-coupling with momentum-dependent coefficient and for big values of p, H_* describes a 2D isotropic oscillator. Other particular interesting cases may be to take $\alpha \propto p^m$. Finally we note that one can begin as well with alternative choices of P_1 and W_1 to generate different isospectral pairs involving Rashba-type Hamiltonians.

Table 1. Five families of isospectral matrix Hamiltonians considered in the main text and associated choices of phase-space functions. In the second and fifth rows the abbreviations $\eta_\pm = (1 \pm k^2)/2$ and $\alpha = \alpha(p)$, $J_3 = xp_y - yp_x$ are used respectively. More general forms and physically relevant special cases of some of these families are given in the main text. The family presented in our previous work [7] constitutes the sixth family of this table.

	Functions	H_*	H_{1F}	H_{2F}
1	$W_1 = 0 = P_1$ $W_2 = \frac{1}{\sqrt{M}}\left(p_1 - \frac{e}{c}A_1\right)$ $P_2 = \frac{1}{\sqrt{M}}\left(p_2 - \frac{e}{c}A_2\right)$	$\frac{1}{2M}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2$	$-\gamma B(\mathbf{q})\sigma_3$	$\gamma B(\mathbf{q})\sigma_3$
2	$W_1 = W_1(\mathbf{q}, \mathbf{p})$ $P_1 = P_1(\mathbf{q}, \mathbf{p})$ $W_2 = kW_1,\ P_2 = kP_1$	$\eta_+(P_1 * P_1 + W_1 * W_1)$	$\mathrm{i}\eta_+[W_1, P_1]_M\sigma_3$	$\mathrm{i}[W_1, P_1]_M(\eta_-\sigma_3 - k\sigma_1)$
3	$W_1 = W_1(x, y)$ $P_1 = \frac{p_x}{\sqrt{M}}$ $W_2 = W_2(x, y)$ $P_2 = \frac{p_y}{\sqrt{M}}$	$\frac{p^2}{2M} + \frac{1}{2}\left(W_1^2 + W_2^2\right)$	$\frac{-\hbar}{2\sqrt{M}}(\partial_x W_1 + \partial_y W_2)\sigma_3$	$\frac{-\hbar}{2\sqrt{M}}(\partial_x W_1 - \partial_y W_2)\sigma_3$ $+ \frac{\hbar}{\sqrt{M}}\partial_x W_2\sigma_1$
4	$W_1 = p_x,\ P_1 = p_y$ $W_2 = u(x, y)$ $P_2 = v(x, y)$	$\frac{p^2}{2} + \frac{1}{4}(u^2 + v^2)$	0	$\hbar[(\partial_y u)\sigma_1 - (\partial_x u)\sigma_2]$
5	$W_1 = \frac{-p_y}{\sqrt{M}},\ P_1 = \frac{p_x}{\sqrt{M}}$ $W_2 = \frac{\alpha\sqrt{M}}{\hbar}J_3$ $P_2 = \frac{\alpha\sqrt{M}}{\hbar}\mathbf{r}\cdot\mathbf{p}$	$\frac{p^2}{2M} + \frac{M}{2}\left[\left(\frac{\alpha r p}{\hbar}\right)^2\right.$ $\left.+ \frac{1}{2}p\alpha'\left(\alpha + \frac{1}{2}p\alpha'\right)\right]$	$\frac{M}{2\hbar}p\alpha\alpha' J_3\sigma_3$	$-H_{1F} + \alpha(\sigma \times \mathbf{p})_3$

6. Concluding remarks

Clifford algebras and their deformations with the Moyal $*$-product have been proved to be profitable in realizing the supersymmetric QM methods on a classical phase space. The method introduced in [7] and elaborated here to reveal its application power addresses the isospectral pairs of 2×2 matrix Hamiltonians depending on four real-valued phase-space functions subject to two conditions given by (5). We have shown that it provides a unified framework for many model Hamiltonians from various branches of physics. Our main results are exhibited in table 1 where the families of the isospectral pairs are tabulated in the order they are considered in the main text. However, our investigation is by no means exhaustive. The resulting structure is quite general to include other families which are mentioned neither here nor in [7]. Even in the discussed families one may identify some physically more relevant special isospectral pairs.

Finally we should emphasize some merits of the method which are not considered here but are quite evident. One of each isospectral pair, namely H_1, is diagonal and in majority of cases its spectrum and eigen-spinors are easily obtainable. The latter can be directly transferred to H_2 by means of L_j's. We should also note that each Hamiltonian can be expressed as a sum of two factorized products of L_j's and their Hermitian conjugates. Another merit of the method is that it directly provides us with constants of motion. Indeed, from (3) it easily follows that $R_j = L_j *_{\mathrm{MC}} L_j^\dagger$ and $S_j = L_j^\dagger *_{\mathrm{MC}} L_j$ are constants of motion such that R_1 and S_2 (R_2 and S_1) commute, with respect to $*_{\mathrm{MC}}$-product, with H_1 (with H_2). However, these are not independent since their sum is proportional to the corresponding Hamiltonian. Therefore, as no explicit time dependence is assumed, each system has, together with the Hamiltonian, two constants of motion. Since H_1 is diagonal, its constants of motion, S_2 and R_1, are simply related to its diagonal elements and they may be important in searching its own SUSY structure. However, constants of motion for H_2 manifest its highly nontrivial symmetry.

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Appendix A. Clifford algebra $\mathbf{C}_4(\mathbb{C})$ and SUSY

Clifford algebra structure can be defined on any vector space on which a non-degenerate inner product g is defined. The non-degeneracy of g provides a linear isomorphism between the vector space and its dual 1-form spaces by $\tilde{x}(y) = i_y(\tilde{x}) = g(x, y)$, where $\tilde{x}$ is 1-form dual to the vector field x and y is an arbitrary vector field. Here i_y stands for the interior derivation (or interior multiplication) which acts on an arbitrary k-form (a totally anti-symmetric covariant tensor) β as $(i_{\tilde{y}}\beta)(x_1, \ldots, x_{k-1}) = k\beta(y, x_1, \ldots, x_{k-1})$, where $x_1, \ldots, x_{k-1}$ are arbitrary vector fields. The associative Clifford product ($*_C$) of a 1-form $\tilde{x}$ and a k-form β can be defined as $\tilde{x} *_C \beta = \tilde{x} \wedge \beta + \mathrm{i}_x\beta$, where $\wedge$ represents the well-known exterior (or Cartan) product. Thanks to the associativity of $*_C$, this relation suffices to completely determine the Clifford product of arbitrary forms [7, 10].

Let us return to the cotangent space of our phase space and let us represent the orthonormal 1-form basis of the complex Clifford algebra $\mathbf{C}_4(\mathbb{C})$ by e^j such that

$$e^j *_C e^k + e^k *_C e^j = 2\delta^{jk}.$$

Here the Kronecker symbols δ^{jk} denote the components of the inverse of g. We then define two Clifford 1-form fields

$$q_- = \bar{C}_1 f + \bar{C}_2 g, \qquad q_+ = C_1 \check{f} + C_2 \check{g}, \tag{A.1}$$

where $\bar{C}_2, \bar{C}_1$ are the complex conjugates of the phase-space functions defined by (5) of the main text and

$$f = \tfrac{1}{\sqrt{2}}(e^1 + \mathrm{i}e^3), \qquad \check{f} = \tfrac{1}{\sqrt{2}}(e^1 - \mathrm{i}e^3),$$
$$g = \tfrac{1}{\sqrt{2}}(e^2 + \mathrm{i}e^4), \qquad \check{g} = \tfrac{1}{\sqrt{2}}(e^2 - \mathrm{i}e^4).$$

By the deformation of Clifford algebra it is meant that the Moyal star-product must be used together with $*_C$ in composing the differential forms. The Moyal–Clifford product $*_{\mathrm{MC}}$ is this combined product. One can directly verify that $q_\pm$ are nilpotent ($q_\pm *_C q_\pm = 0$) with respect to the Clifford product. On the other hand they obey $q_\pm *_{\mathrm{MC}} q_\pm = 0$, if and only if $[C_1, C_2]_M = 0$ is satisfied [7]. This single condition is equivalent to two conditions (5) of the main text. We then define the supersymmetric Hamiltonian H_s by $2H_s = \{q_+, q_-\}_{\mathrm{MC}}$ which implies the commutativity, with respect to $*_{\mathrm{MC}}$-product, of H_s with $q_\pm$. Therefore, the set $\{H_s, q_\pm\}$ closes into the so-called 2-extended (or $N = 2$) SUSY algebra with the nilpotent supercharges $q_\pm$.

By virtue of (A1), H_s can explicitly evaluated to be

$$\begin{aligned} H_s = H_*\mathbb{I} + \tfrac{1}{2}\{&[W_1, P_1]_M e^1 *_C e^3 + [W_2, P_2]_M e^2 *_C e^4 \\ &+ [W_2, P_1]_M (e^1 *_C e^4 + e^2 *_C e^3) + [W_1, W_2]_M (e^1 *_C e^2 + e^3 *_C e^4)\}, \end{aligned} \tag{A.2}$$

where $\mathbb{I}$ stands for the unit element of the algebra. If the representation

$$e^1 = \begin{pmatrix} 0 & \mathrm{i}\sigma_1 \\ -\mathrm{i}\sigma_1 & 0 \end{pmatrix}, \qquad e^2 = \begin{pmatrix} 0 & \mathrm{i}\sigma_3 \\ -\mathrm{i}\sigma_3 & 0 \end{pmatrix},$$
$$e^3 = \begin{pmatrix} 0 & \mathrm{i}\sigma_2 \\ -\mathrm{i}\sigma_2 & 0 \end{pmatrix}, \qquad e^4 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},$$

is used for the complex Clifford basis, then the MC-product goes over to the Moyal product of matrix-valued functions. By using the above representation in (A2), we end up with the block diagonal form $H_s = \mathrm{diag}(H_1, H_2)$, where H_j's are given in the introduction of the main text. In this representation, the supercharges $q_\pm$ are

$$q_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & L_1 \\ -L_2 & 0 \end{pmatrix} = q_-^\dagger.$$

Nilpotency of $q_\pm$ imply that L_j's are divisors of zero ($L_1 *_{\mathrm{MC}} L_2 = 0 = L_2 *_{\mathrm{MC}} L_1$) and the $*_{\mathrm{MC}}$ commutativity of H_s and $q_\pm$ is equivalent to relations (3) of the main text. Finally, as $\omega_1 = q_+ + q_-$ and $\omega_2 = -\mathrm{i}(q_+ - q_-)$ anti-commute with respect to the $*_{\mathrm{MC}}$-product and become Hermitian in the above representation, we can write

$$4H_s\delta_{jk} = \{\omega_j, \omega_k\}_{\mathrm{MC}}, \qquad [H_s, \omega_j]_{\mathrm{MC}} = 0.$$

These constitute a realization of the above SUSY algebra in terms of Hermitian supercharges ω_j.

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